Rotating version of the static R=0 Lorentzian wormhole: geometry, matter and shadow

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GR24 & Amaldi16

A static wormhole with zero Ricci scalar (R=0)

$$ds^2 = -(p+1)^{-2}\left(p+\sqrt{1-\frac{2M}{r}}\right)^2 dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Throat radius at $r = 2M$
- Ricci scalar, R=0
- Static, spherically symmetric and asymptotically flat Lorentzian wormhole
- Required matter violates energy conditions (assuming GR)

Phys. Rev. D 65, 064004 (2002), Phys. Rev. D 65, 084040 (2002).

Newman-Janis (NJ) algorithm and its failure

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{g(r)} + h(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

- I: Rewrite the metric in Eddington-Finkelstein coordinates
- II: Decomposition of inverse metric using a chosen null tetrad
- III: Complex transformation: $r \to r + ia\cos\theta$, $u \to u - ia\cos\theta$
  - Metric functions transforms: $r^2 \to rr^*$, $\frac{2}{r} \to \frac{1}{r} + \frac{1}{r^*}$
- IV: Transformation to Boyer-Lindquist coordinate

Step IV fails: No global coordinate transformation exists for the given metric.

J. Math. Phys. (N.Y.) 6, 915 (1965).

The Azreg-Aïnou (AA) method

- First two steps are same as NJ algorithm.

Difference comes in the step III

Assumptions under complex transformation:

$$f(r), g(r), h(r) \to A(r,\theta,a), B(r,\theta,a), \Psi(r,\theta,a)$$

$$\lim_{a\to 0} A(r,\theta,a) = f(r), \quad \lim_{a\to 0} B(r,\theta,a) = g(r), \quad \lim_{a\to 0}\Psi(r,\theta,a) = h(r).$$

No need of complexification

- The step IV is performed through: $du = dt - \frac{h\sqrt{g/f}+a^2}{gh+a^2}dr, \quad d\phi = d\varphi - \frac{a}{gh+a^2}dr$
- The three unknows (A, B, Ψ) of the new rotating metric fixed through: $g_{tr} = 0, \quad g_{r\phi} = 0$ and $G_{r\theta} = 0$
- At, $a \to 0$, $ds^2_{rotating} = \sqrt{g/f}\, ds^2_{static}$ (**extra conformal factor**) Phys. Letts. B 730, 95–98 (2014).

The new rotating wormhole using AA method

- To remove the conformal factor, we start with the following transformed static metric: $\mathrm{d}s^2 = -f(\ell)\mathrm{d}t^2 + \frac{\mathrm{d}\ell^2}{f(\ell)} + h(\ell)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$
- Applying all steps of AA method, we have the rotating metric in ℓ, converting ℓ to r,

$$\Delta(r) = \left(1-\frac{2M}{r}\right)\left[\frac{\left(p+\sqrt{1-2M/r}\right)^2 r^2 + a^2(p+1)^2}{\left(p+\sqrt{1-2M/r}\right)^2}\right]$$

$$m(r) = \frac{r}{2}\left[1 - \frac{\left(p+\sqrt{1-2M/r}\right)^2}{(p+1)^2}\right]$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

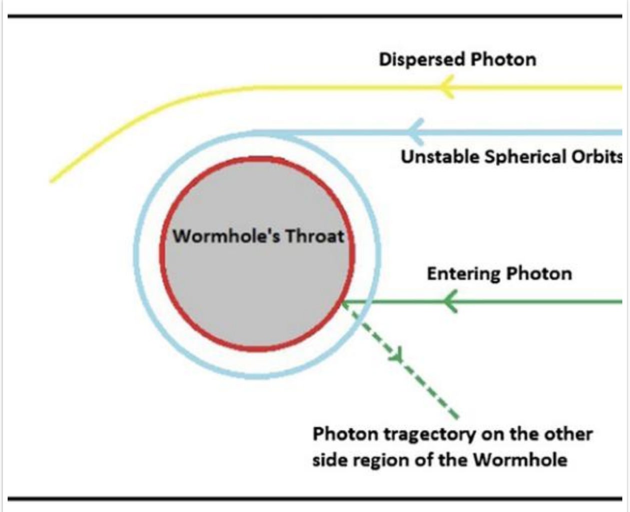
$$\mathrm{d}s^2 = -\left[1-\frac{2m(r)r}{\Sigma}\right]\mathrm{d}t^2 - \frac{4m(r)ra\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \sin^2\theta\left[r^2 + a^2 + \frac{2m(r)ra^2\sin^2\theta}{\Sigma}\right]\mathrm{d}\varphi^2$$

- p = 0 → Kerr black hole
- a = 0 → static wormhole
- The shape function satisfies all the requirements of wormhole
- The required matter violates the energy conditions (assuming GR)
- All independent curvature scalars are nonsingular everywhere.

Phys. Rev. D 111, 064010 (2025)

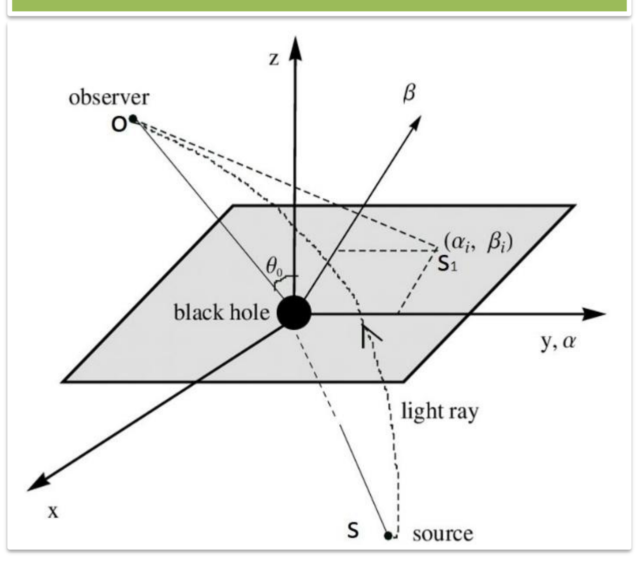
Shadow profiles and observations

Wormhole shadow



- Photons can fall in the wormhole or scatter.
- The boundary between infalling and outgoing is photon sphere.

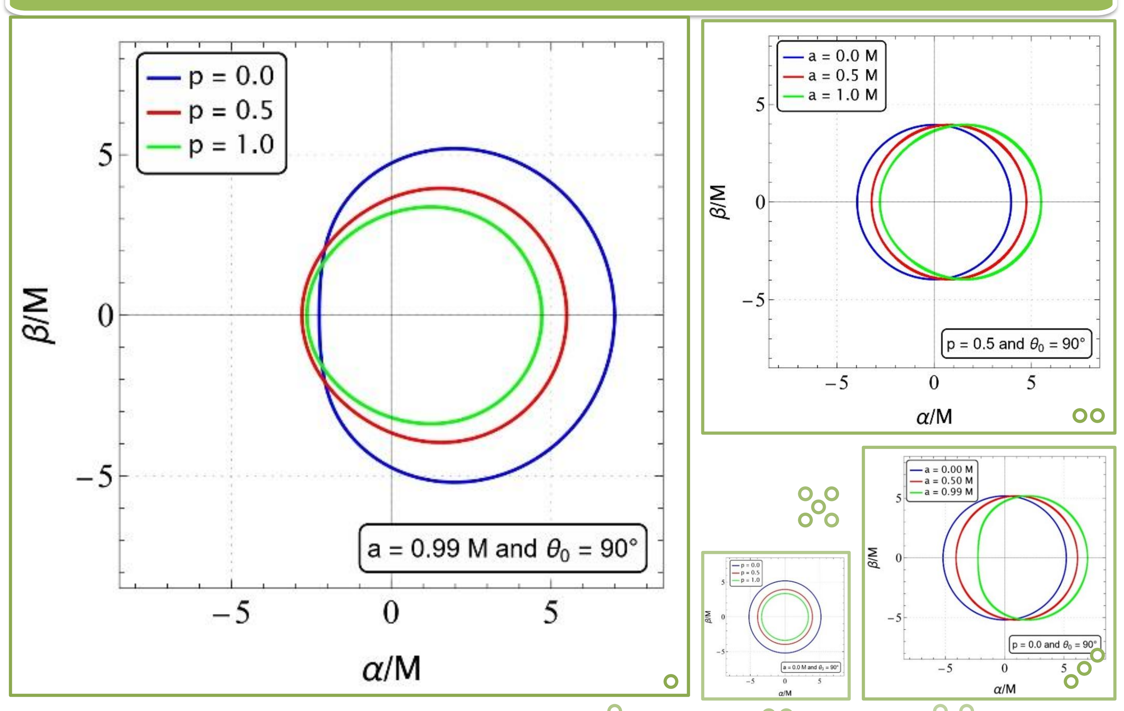
Celestial coordinates



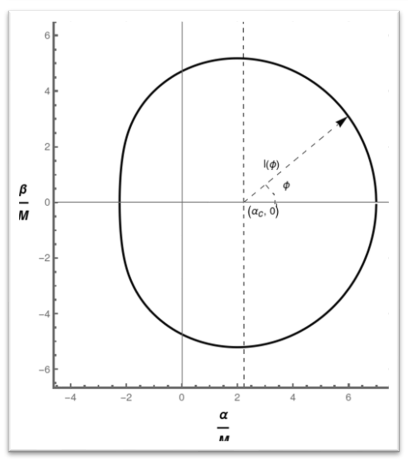
$$\alpha = \lim_{r_0\to\infty} -r_0^2 \sin\theta_0 \left.\frac{d\phi}{dr}\right|_{r_0}$$

$$\beta = \lim_{r_0\to\infty} r_0^2 \left.\frac{d\theta}{dr}\right|_{r_0}$$

Theoretical shadow profiles



Observable quantities



- Deviation from circularity:

$$\Delta C = \frac{1}{R_{avg}}\sqrt{\frac{1}{2\pi}\int_0^{2\pi} d\phi\,(\ell(\phi) - R_{avg})^2}$$

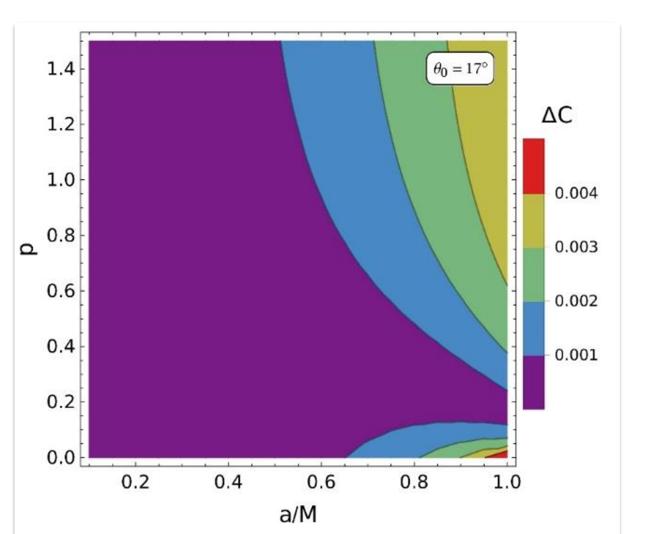
- Fractional deviation parameter:

$$\delta = \frac{d_{sh}}{d_{sh,Sch}} - 1 = \frac{R_{avg}}{3\sqrt{3}M} - 1$$

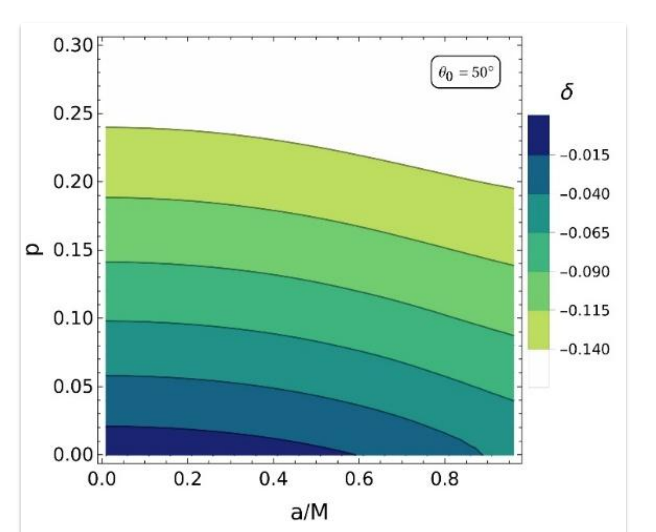
$$R^2_{avg} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\,\ell^2(\phi)$$

EHT results and comparison with the metric

- The observed angular diameter for M87* is $42 \pm 3\ \mu as$, and for SgrA* it is $51.8 \pm 2.3\ \mu as$. APJ. Lett. 875, L1 (2019), APJ. Lett. 930, L12 (2022).
- For M87* $\Delta C \lesssim 10\%$ APJ. Lett. 875, L5 (2019)
- For SgrA* $\delta = \begin{cases} -0.08^{+0.09}_{-0.09} & \text{(VLTI)} \\ -0.04^{+0.09}_{-0.10} & \text{(Keck)} \end{cases}$ APJ. Lett. 930 L17 (2022)



The contour plot of ΔC at $\theta_0 = 17^0$



The contour plot of δ at $\theta_0 = 50^0$

- **All values of p respect the restriction from M87***
- $p < 0.24$ **is required for Sgr A***

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Reference: A. Kar, S. Jana, S. Kar, Phys. Rev. D **111**, 064010 (2025)